

TOROIDAL Z -ALGEBRAS

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Introduction

The purpose of this paper is to construct faithful representation of twisted toroidal Lie algebra of type ADE. Twisted toroidal Lie algebras are universal central extensions of $N + 1$ twisted multiloop algebras. If $N = 0$ they are precisely the twisted affine Kac-Moody Lie algebras. In the process we recover the main results of [EM], [MEY] (homogeneous picture), [B1] and [T] (principal picture) for the non-twisted case. These works have applications in the solutions of differential equations. See [B2], [ISW1] and [ISW2]. We expect similar applications for the twisted case also. Though the idea is the same our proofs are much shorter than those in the above works. The paper can be seen as a generalisation of certain results of [LW].

Let \mathcal{G} be the simple finite dimensional Lie algebra over the complex numbers. Let A be a Laurent polynomial ring in $N + 1$ commuting variables.

Consider the multiloop algebra $\mathcal{G} \otimes A$, its universal central extension $\tilde{\tau}$ the toroidal Lie algebra. Let θ be an automorphism of \mathcal{G} of order m . Then θ can be extended to an automorphism of $\tilde{\tau}$ (Section 1). Then the subalgebra of θ fixed points inside $\tilde{\tau}$ is called twisted toroidal Lie algebra $\tilde{L}(\mathcal{G}, \theta)$. It is the universal central extension of the underlining multiloop algebra (See [BK]).

In Section 1, we define a category \underline{C}_k of $\tilde{L}(\mathcal{G}, \theta)$ modules which satisfy a factorisation property first introduced in [BY]. The factorisation property is not satisfied for a general class of integrable modules. But there are enough of integrable modules which satisfy the factorisation property. For example the vertex representation defined in [EM] and the representation considered in [BY] satisfy factorisation property.

Next by following [LW] closely we define toroidal Z -algebras (1.10) and define a category \underline{D}_k -of Z -algebra modules. We then prove the important Proposition (2.6) which says that the categories \underline{C}_k and \underline{D}_k are equivalent. Thus by constructing a Z -algebra module we get a module for $\tilde{L}(\mathcal{G}, \theta)$.

In section 3 we specialise to the homogeneous picture for the nontwisted case of type ADE. We construct a module for the Z -toroidal Lie algebra closely following the results of [LP]. Thereby constructing a module for $\tilde{L}(\mathcal{G}, I_d) \cong \tilde{\tau}$ which is faithful. This recovers the main result of [EM]. Our calculations are certainly much shorter.

In Section 4 we specialise to the principal picture. This includes the twisted and nontwisted toroidal Lie algebras. We again construct a module for the Z -toroidal Lie algebra by making use of the corresponding results for the affine Kac-Moody Lie algebra from [LW]. We have to consider the additional Fock space for this purpose. Thus we get a module for our $\tilde{L}(\mathcal{G}, \theta)$. This result recovers the main result of [B1] and [T]. Again our proof are much shorter. The twisted case is completely new.

In the process we have given the following realization of twisted toroidal Lie-algebra. Let π be a Dynkin diagram automorphism of \mathcal{G} . Define an

automorphism θ of \mathcal{G} as in the section 4. Then we prove that $\tilde{\tau}(\mathcal{G}, \theta) \cong \tilde{L}(\mathcal{G}, \pi)$. This is what is called the principal realization in the affine case. The isomorphism is given explicitly in twisted case and it is completely new even in the affine case (Proposition 4.10).

Section 1

Let \mathcal{G} be a finite dimensional semisimple Lie-algebra over the complex numbers \mathbb{C} . Let $<, >$ be a non-degenerate symmetric \mathcal{G} -invariant bilinear form on \mathcal{G} . We fix a nonnegative integer N . Let $A = \mathbb{C}[t^\pm, t_1^\pm, \dots, t_N^\pm]$ be the ring of Laurent polynomials in $N+1$ commuting variables. Let $\underline{r} = (r_1, \dots, r_N) \in \mathbb{Z}^N$. Let $t^{\underline{r}} = t_1^{r_1} t_2^{r_2} \dots t_N^{r_N}$. Fix a positive integer m . Let Ω_A be a free A -module of rank $N+1$ with basis $\{k_0, \dots, k_N\}$. Let d_A be the subspace of Ω_A spanned by elements of the form $\frac{1}{m} r_0 t^{r_0} t^{\underline{r}} k_0 + \dots + r_N t^{r_0} t^{\underline{r}} k_N$. Let $x(r_0, \underline{r}) = x \otimes t^{r_0} t^{\underline{r}} \epsilon \mathcal{G} \otimes A$. Then the toroidal Lie-algebra $\tau = \mathcal{G} \otimes A \oplus \Omega_A / d_A$ is defined by the following bracket.

$$[x(r_0, \underline{r}), y(s_0, \underline{s})] = [x, y](r_0 + s_0, \underline{r} + \underline{s}) + \frac{\langle x, y \rangle}{m} r_0 t^{r_0 + s_0} t^{\underline{r} + \underline{s}} k_0 + \langle x, y \rangle \sum_{i=1}^N r_i t^{r_0 + s_0} t^{\underline{r} + \underline{s}} k_i$$

for $x, y \in \mathcal{G}, \underline{r}, \underline{s} \in \mathbb{Z}^N, r_0, s_0 \in \mathbb{Z}$.

Ω_A / d_A is central.

It is known that τ is the universal central extension of $\mathcal{G} \otimes A$. (See [K], [MEY]). (First note that the toroidal Lie algebra defined by $m = 1$ is isomorphic to the above). Let \underline{h} be a Cartan subalgebra of \mathcal{G} . Let θ be an automorphism of \mathcal{G} such that $\theta(\underline{h}) = \underline{h}$ and of order m . Assume that the form $<, >$ is θ -invariant that is $\langle \theta x, \theta y \rangle = \langle x, y \rangle$. Let $\mathbb{Z}_m = \mathbb{Z} / m\mathbb{Z}$ be the cyclic group of order m . Let w be a primitive m th root of unity.

(1.2) Let $\mathcal{G}_i = \{x \in \mathcal{G} \mid \theta x = w^i x\}$ for $i \in \mathbb{Z}$. Then $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}_m} \mathcal{G}_i$. Note that $\langle \mathcal{G}_i, \mathcal{G}_j \rangle = 0$ unless $i + j \equiv 0(m)$. For $x \in \mathcal{G}$ write $x = \sum_{i \in \mathbb{Z}_m} x_i$ where $\theta x_i = w^i x_i$. Define $x_i = x_{\bar{i}}$ for $i \in \mathbb{Z}$ and $\bar{i} \in \mathbb{Z}_m$.

Extend the automorphism θ to τ by $\theta(x(r_0, \underline{r})) = w^{-r_0} \theta(x)(r_0, \underline{r})$ and $w(t_0^{r_0} t^{\underline{r}} k_i) = w^{-r_0} t^{r_0} t^{\underline{r}} k_i, 0 \leq i \leq N$. Let $\tilde{\tau} = \tau \oplus D$ where D is spanned by derivations $\{d_0, \dots, d_N\}$ with bracket $[d_i, x(r_0, \underline{r})] = r_i x(r_0, \underline{r})$, for $0 \leq i \leq N$. Extend the automorphism θ to $\tilde{\tau}$ by $\theta(d_i) = d_i$. Let $(\Omega_A/d_A)_0$ be the linear span of $t^{r_0} t^{\underline{r}} k_i$ where $r_0 \equiv 0(m)$. Consider the θ fixed points of $\tilde{\tau}$ say $\tilde{L}(\mathcal{G}, \theta)$.

(1.3) . Let $L(\mathcal{G}, \theta) = \bigoplus_{\substack{i \in \mathbb{Z} \\ \underline{r} \in \mathbb{Z}^N}} \mathcal{G}_i(i, \underline{r})$ and $\overline{L}(\mathcal{G}, \theta) = L(\mathcal{G}, \theta) \oplus (\Omega_A/d_A)_0$. Then clearly $\tilde{L}(\mathcal{G}, \theta) = \overline{L}(\mathcal{G}, \theta) \oplus D$.

Since \langle, \rangle is non-degenerate and \mathcal{G} -invariant, its restriction to \underline{h} is also non-degenerate. We identify \underline{h} and \underline{h}^* via this form. Let ϕ be the root system of \mathcal{G} . For $\beta \in \phi$, choose the corresponding non-zero root vectors x_β such that $[x_\beta, x_{-\beta}] = \langle x_\beta, x_{-\beta} \rangle \beta$. Let $\epsilon(\beta, \gamma)$ be a non-zero number such that $[x_\beta, x_\gamma] = \epsilon(\beta, \gamma) x_{\beta+\gamma}$. Clearly the set of roots ϕ is θ -stable. Then define $\eta(p, \beta)$ a non-zero scalar such that

(1.4) $\theta^p x_\beta = \eta(p, \beta) x_{\theta^p \beta}$. For any vector space V and for indeterminates $\zeta_1, \dots, \zeta_\ell$, denote $V\{\zeta_1, \dots, \zeta_\ell\}$ the space of formal Laurent series. Further $V[\zeta_1^{\pm 1}, \dots, \zeta_n^{\pm 1}]$ denote finite formal Laurent series. We recall the following Proposition from [LW]. Definition $\delta(\zeta) = \sum_{i \in \mathbb{Z}} \zeta^i \in \mathbb{C}\{\zeta\}$.

Proposition (1.5) (a) (Proposition (2.2) of [LW]). Let $f(\zeta) \in V[\zeta, \zeta^{-1}]$. Then

$$f(\zeta) \delta(\zeta^m) = \sum_{p \in \mathbb{Z}_m} f(w^p) \delta(w^{-p} \zeta).$$

For $x \in \mathcal{G}, \underline{r} \in \mathbb{Z}^N$ let

$$x(\underline{r}, \zeta) = \sum_i x_i \otimes t^i t^{\underline{r}} \zeta^i, \quad k_i(\underline{r}, \zeta^m) = \sum_p t^{mp} t^{\underline{r}} k_i \zeta^{mp}.$$

For any infinite series $f(\zeta) = \sum b_i \zeta^i$, let $Df\zeta = \sum i b_i \zeta^i$.

Proposition (1.5) The following relations hold for $x(\underline{r}, \zeta)$ and $k_0(\underline{r}, \zeta^m)$. In fact they define a Lie-algebra $\tilde{L}(\mathcal{G}, \theta)$. For $\beta_1, \beta_2 \in \phi, \underline{r}, \underline{s} \in \mathbb{Z}^N$.

$$\begin{aligned} (1) \quad & [x_{\beta_1}(\underline{r}, \zeta_1), x_{\beta_2}(\underline{s}, \zeta_2)] \\ &= \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \phi} \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) X_{\theta^p \beta_1 + \beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ & - \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \beta_2(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ & + \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \left(\sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \right. \\ & \quad \left. + \frac{k_0(\underline{r} + \underline{s}, \zeta_2^m)}{m} D\delta(w^{-p} \zeta_1 / \zeta_2) \right) \end{aligned}$$

$$\begin{aligned} (2) \quad & [\beta_1(\underline{r}, \zeta_1), \beta_2(\underline{s}, \zeta_2)] \\ &= \frac{1}{m} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle \left(\sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \right. \\ & \quad \left. + \frac{k_0(\underline{r} + \underline{s}, \zeta_2^m)}{m} D\delta(w^{-p} \zeta_1 / \zeta_2) \right). \end{aligned}$$

$$\begin{aligned} (3) \quad & [\beta_1(\underline{r}, \zeta_1), x_{\beta_2}(\underline{s}, \zeta_2)] \\ &= \frac{1}{m} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle x_{\beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \end{aligned}$$

$$(4) \quad \frac{1}{m} Dk_0(\underline{x}, \zeta^m) + \sum_{i=1}^N r_i k_i(\underline{x}, \zeta^m) = 0$$

$$(5) \quad [d_i, k_j(\underline{x}, \zeta^m)] = r_i k_j(\underline{x}, \zeta^m) \text{ for } 1 \leq i \leq N \text{ and } 0 \leq j \leq N$$

$$(6) \quad [d_0, k_j(\underline{x}, \zeta^m)] = Dk_j(\underline{x}, \zeta^m), \quad 0 \leq j \leq N$$

$$(7) \quad x_\beta(\underline{x}, w^p \zeta) = \eta(p, \beta) x_{\theta^p \beta}(\underline{x}, \zeta)$$

$$(8) \quad k_i(\underline{x}, \zeta^m) \text{ is central for } 0 \leq i \leq N.$$

Proof (4) follows from the definition of $(\Omega_A/d_A)_0$. (5), (6), (7) and (8) are easy to see. First consider the following:

$$[x(\underline{x}, \zeta_1), y(\underline{s}, \zeta_2)] = F + G_1 + G_2.$$

Where

$$F = \sum_{i,j} [x_i, y_j] t_0^{i+j} t^{\underline{x}+\underline{s}} \zeta_1^i \zeta_2^j,$$

$$G_1 = \frac{1}{m} \sum_{i,j} i < x_i, y_j > t_0^{i+j} t^{\underline{x}+\underline{s}} k_0 \zeta_1^i \zeta_2^j,$$

$$G_2 = \sum_{\ell=1}^N \sum_{i,j} r_\ell i < x_i, y_j > t_0^{i+j} t^{\underline{x}+\underline{s}} k_\ell \zeta_1^i \zeta_2^j.$$

From the proof of Theorem (2.3) of [LW] it following that

$$F = \frac{1}{m} \sum_{p \in \mathbb{Z}_m} [\theta^p x, y](\underline{x} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2).$$

Now consider $[\theta^p x_{\beta_1}, x_{\beta_2}] = \eta(p, \beta_1) [x_{\theta^p \beta_1}, x_{\beta_2}]$.

$$= \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) x_{\theta^p(\beta_1 + \beta_2)} \text{ if } \theta^p \beta_1 + \beta_2 \in \Phi$$

$= -\eta(p, \beta_1) < x_{\beta_2}, x_{-\beta_2} > \beta_2$ if $\theta^p \beta_1 + \beta_2 = 0$
 $= 0$ if $\theta^p \beta_1 + \beta_2 \neq 0$ and not a root.

Thus for $x = x_{\beta_1}$ and $y = x_{\beta_2}$, F equals to the first and second term of RHS in (1). For G_1 and G_2 , first note that $< x_i, y_j > = 0$ if $i + j \neq 0(m)$ by (1.2). Thus

$$\begin{aligned}
mG_1 &= \sum_{i,p} i < x_i, y_{-i+mp} > t_0^{mp} t^{\underline{r}+\underline{s}} k_0(\zeta_1/\zeta_2)^i \zeta_2^{mp}. \\
&= \sum_{i,p} i < x_i, y_{-i} > t_0^{mp} t^{\underline{r}+\underline{s}} k_0(\zeta_1/\zeta_2)^i \zeta_2^{-mp} \\
&= \sum_{i,p} i < x_i, y > t_0^{mp} t^{\underline{r}+\underline{s}} k_0(\zeta_1/\zeta_2)^i \zeta_2^{mp} \\
&= D\left(\sum_i < x_i, y > (\zeta_1/\zeta_2)^i\right) k_0(\underline{r} + \underline{s}, \zeta_2^m) \\
&= D\left(\sum_{i=0}^{m-1} < x_i, y > (\zeta_1/\zeta_2)^i \delta((\zeta_1/\zeta_2)^m) \cdot k_0(\underline{r} + \underline{s}, \zeta_2^m)\right) \\
&= \frac{1}{m} \sum_p < \theta^p x, y > D\delta(w^{-p} \zeta_1/\zeta_2) k_0(\underline{r} + \underline{s}, \zeta_2^m). \tag{1.5(1)}
\end{aligned}$$

Similarly

$$G_2 = \sum_{i=1}^N r_i < \theta^p x, y > \delta(w^{-p} \zeta_1/\zeta_2) k_\ell(\underline{r} + \underline{s}, \zeta_2^m) \tag{1.5(2)}$$

Again for $x = x_{\beta_1}$ and $y = x_{\beta_2}$, $< \theta^p x, y > = \eta(p, \beta_1) < x_{\theta^p \beta_1}, x_{\beta_2} > = 0$ if $\theta^p \beta_1 + \beta_2 \neq 0$

$$= \eta(p, \beta_1) < x_{-\beta_2}, x_{\beta_2} > \text{ if } \theta^p \beta_1 + \beta_2 = 0.$$

This completes the proof (1). To see (2), take $x = \beta_1$ and $y = \beta_2$. Then $F = 0$. From (1.5 (1) and 1.5(2), (2) will follow. To see (3) take $x = \beta_1$ and $y = x_{\beta_2}$ and note that $G_1 = 0$ and $G_2 = 0$ and F is equal to RHS of 3. This completes the proof of the Proposition (1.5).

Now we define a category \underline{C}_k of $\tilde{L}(\mathcal{G}, \theta)$ - modules.

Definition (1.6) A $\tilde{L}(\mathcal{G}, \theta)$ -module V is in $\underline{\mathcal{C}}_k$ if

- (1) k_o acts as k .
- (2) $V = \oplus_{z \in \mathbb{C}} V_z, V_z = \{v \in V \mid d_0 v = zv\}$. Assume for any z there exists $\ell_0 \ni V_{z+\ell} = 0$ for $\ell > \ell_0$
- (3) $\frac{1}{k} X(\underline{r}, \zeta) k_0(\underline{s}, \zeta^m) = X(\underline{r} + \underline{s}, \zeta)$
 $\frac{1}{k} k_i(\underline{r}, \zeta^m) k_o(\underline{s}, \zeta^m) = k_i(\underline{r} + \underline{s}, \zeta^m)$ for $0 \leq i \leq N$.

Remark (1.7) Condition (3) is not satisfied for most of the modules. But there are enough of them which are sufficient for a realization of $L(\mathcal{G}, \theta)$. For examples vertex operator representation of [EM] satisfy the condition (3) as well as the representations considered in [BY].

(1.8) Consider the Lie subalgebra of $\tilde{L}(\mathcal{G}, \theta), \tilde{\underline{h}} = \oplus_{i \in \mathbb{Z}} h_i \otimes t^i \oplus \mathbb{C} k_0 \oplus \mathbb{C} d_0(h \in \underline{h})$. The bracket is given by

$$[h_i \otimes t^i, h_j \otimes t^j] = ik_0 \langle h_i, h_j \rangle \delta_{i+j,0}.$$

Clearly $\tilde{\underline{h}}$ is \mathbb{Z} -graded. Let $M(k)$ be a Verma module of level k for $\tilde{\underline{h}}$. Then it is a standard fact that $M(k)$ is irreducible whenever k is non-zero.

Proposition (1.9) Any module V in $\underline{\mathcal{C}}_k (k \neq 0)$ has the following decomposition as $\tilde{\underline{h}}$ -modules. $V \cong M(k) \otimes \Omega_V$ where

$$\Omega_V = \{v \in V \mid h_i \otimes t^i v = 0 \text{ for } i > 0\}$$

see Proposition 5.4 of [LW].

We now define toroidal Z algebras. Notation as earlier. For $\alpha \in \phi, \underline{r} \in \mathbb{Z}^N$ let $Z(\alpha, \underline{r}, \zeta)$ be a series in ζ with integral powers. For $\underline{r} \in \mathbb{Z}^N$ let $k_i(\underline{r}, \zeta^m)$ be a series in ζ^m . The toroidal Z_k -algebra or simply Z_k -algebra is spanned by the components of $Z(\alpha, \underline{r}, \zeta), k_i(\underline{r}, \zeta^m)$ and \underline{h}_0 , by the following relation.

(1.10) **Relations** $\alpha, \beta, \beta_1, \beta_2 \in \Phi, \underline{r}, \underline{s} \in \mathbb{Z}^N$.

$$(1) \quad \frac{1}{k} Z(\alpha, \underline{r}, \zeta) k_0(\underline{s}, \zeta^m) = Z(\alpha, \underline{r} + \underline{s}, \zeta)$$

$$(2) \quad \frac{1}{k} k_o(\underline{r}, \zeta^m) k_i(\underline{s}, \zeta^m) = k_i(\underline{s} + \underline{r}, \zeta^m), \quad 0 \leq i \leq N.$$

$$(3) \quad \sum_{i=1}^N r_i k_i(\underline{r}, \zeta^m) + \frac{1}{m} D k_0(\underline{r}, \zeta^m) = 0.$$

$$(4) \quad [d_0, Z(\beta, \underline{r}, \zeta)] = D Z(\beta, \underline{r}, \zeta)$$

$$(5) \quad [d_0, k_i(\underline{r}, \zeta^m)] = D k_i(\underline{r}, \zeta^m), \quad 0 \leq i \leq N$$

$$(6) \quad [d_i, k_j(\underline{r}, \zeta^m)] = r_i k_j(\underline{r}, \zeta^m), \quad 1 \leq i \leq N, 0 \leq j \leq N$$

$$\begin{aligned} (7) \quad & \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{<\theta^p \beta_1, \beta_2>} Z(\beta_1, \underline{r}, \zeta_1) Z(\beta_2, \underline{s}, \zeta_2) \\ & - \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_2 / \zeta_1)^{<\theta^p \beta_1, \beta_2>} Z(\beta_2, \underline{s}, \zeta_2) Z(\beta_1, \underline{r}, \zeta_1) \\ & = \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \phi} \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) Z(\theta^p \beta_1 + \beta_2, \underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ & - \frac{1}{mk} <x_{\beta_2}, x_{-\beta_2}> \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) (\beta_2)_0 k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \\ & + \frac{1}{m} <x_{\beta_2}, x_{-\beta_2}> \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \left(\sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) + \right. \\ & \quad \left. \frac{1}{mk} k_0(\underline{r} + \underline{s}, \zeta_2) D \delta(w^{-p} \zeta_1 / \zeta_2) \right). \end{aligned}$$

$$(8) \quad [\alpha, Z(\beta, \underline{r}, \zeta)] = <\alpha, \beta> Z(\beta, \underline{r}, \zeta), \quad \alpha \in \underline{h}_0$$

$$(9) \quad Z(\beta, \underline{r}, \zeta) = \eta(p, \beta) Z(\theta^p \beta, \underline{r}, \zeta)$$

$$(10) \quad k_i(\underline{r}, \zeta^m) \text{ is central for } 0 \leq i \leq N.$$

As it is we do not know whether a Z_k algebra is non-zero or not but certainly it is well defined.

(1.11) Definition A Z_k module W is said to be in the category \underline{D}_k if

- (1) k_0 acts by scalar k .
- (2) $V = \oplus_{z \in \mathbb{C}} V_z$, $V_z = \{v \in V \mid d_0 v = zv\}$. Assume for any given z there exists $\ell_0 \ni \ell > \ell_0$ implies $V_{z+\ell} = 0$.

Section 2

In this section we establish equivalence between the categories \underline{C}_k and \underline{D}_k . The proof are very similar to [LW]. In fact most of the results go through.

Let $V \in \underline{C}_k$. Define for $\beta \in \phi$.

$$E^\pm(\beta, \zeta) = \exp(\pm m \sum_{J>0} \beta_i \otimes t^i \zeta^{\pm j} / jk)$$

and $Z(\beta, \underline{r}, \zeta) = E^-(\beta, \zeta) x_\beta(\underline{r}, \zeta) E^+(\beta, \zeta)$. We first prove that these Z -operators satisfy relations in (1.10).

We first recall the following from section 3 of [LW]. We are taking $\underline{a} = \underline{h}$ and $\underline{m} = 0$ (in decomposition (3.5)) in [LW].

Proposition (2.1) $\alpha, \beta, \gamma \in \Phi$

$$(1) \quad i > 0$$

$$(a) \quad [\alpha_i \otimes t^i, E^+(\beta, \zeta)] = 0$$

$$(b) \quad [\alpha_{-i} \otimes t^{-i}, E^-(\beta, \zeta)] = 0$$

$$(c) \quad [\alpha_i \otimes t^i, E^-(\beta, \zeta)] = - \langle \alpha_i, \beta \rangle \zeta^{-i} E^-(\beta, \zeta).$$

$$(d) \quad [\alpha_{-i} \otimes t^{-i}, E^+(\beta, \zeta)] = - \langle \alpha_i, \beta \rangle \zeta^i E^+(\beta, \zeta)$$

$$(2) \text{ (a) } E^\pm(\beta + \gamma, \zeta) = E^\pm(\beta, \zeta), E^\pm(\gamma, \zeta)$$

$$(b) E^\pm(\theta^p \beta, \zeta) = E^\pm(\beta, w^p \zeta)$$

$$(c) DE^\pm(\beta, \zeta) = \frac{m}{k} \beta(\zeta)^\pm E^\pm(\beta, \zeta)$$

where $\beta(\zeta)^\pm = \sum \beta_\pm \otimes t^{\pm 1} \zeta^{\pm i}$

$$(3) \text{ (a) } x_\beta(\underline{r}, \zeta) = E^-(-\beta, \zeta) Z(\beta_1, \underline{r}, \zeta) E^+(-\beta, \zeta)$$

$$(b) Z(\beta, \underline{r}, w^p \zeta) = \eta(p, \beta) Z(\theta^p \beta, \underline{r}, \zeta)$$

$$(4) E^+(\beta, \zeta_1) E^-(\gamma, \zeta_2) = E^-(\gamma, \zeta_2) E^+(\beta, \zeta_1).$$

$$(5) \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{\frac{<\theta^p \beta, \gamma>}{k}}$$

$$(6) E^+(\beta, \zeta_1) x_\gamma(\underline{r}, \zeta_2) = x_\gamma(\underline{r}, \zeta_2) E^+(\beta, \zeta_1)$$

$$(7) \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{\frac{-<\theta^p \beta, \gamma>}{k}}$$

$$(8) x_\beta(\underline{r}, \zeta_1) E^-(\gamma, \zeta_2) = E^-(\gamma, \zeta_2) x_\beta(\underline{r}, \zeta_1)$$

$$\prod_{p \in \mathbb{Z}_m} (1 - w^p \zeta_1 / \zeta_2)^{\frac{-<\theta^p \beta, \gamma>}{k}}$$

Proof (1) a and b follows from the definition. To see 1(c) consider

$$\begin{aligned} & [\alpha_i \otimes t^i, -\sum_{j < 0} \beta_j \otimes t^j / j_k \zeta^j] \\ &= -\frac{1}{i} \langle \alpha_i, \beta_{-i} \rangle \zeta^{-i}. \end{aligned}$$

Now

$$\begin{aligned} & [\alpha_i \otimes t^i, \frac{(-\sum \beta_j \otimes t^j / j_k \zeta^j)^\ell}{\ell!}] \\ &= -\frac{\langle \alpha_i, \beta_{-i} \rangle}{(\ell-1)!} (-\sum \frac{\beta_j \otimes t^j}{j_k} \zeta^j)^{\ell-1}. \end{aligned}$$

First note that $\langle \alpha_i, \beta_{-i} \rangle = \langle \alpha, \beta \rangle$ by (1.2). Now 1(c) follows from the definition.

(1)(d) follows from similar argument.

(2) and (3) follows from definition. See also Proposition 3.2 and 3.3 of [LW].

(4) follows from Proposition 3.4 of [LW].

(5) follows from similar argument of Proposition 3.5 of [LW].

Corollary (2.2) Let $\tilde{h}^1 = \oplus_{i \neq 0} \underline{h}_i \otimes t^i$. Then as operators the following hold. (1) $[\tilde{h}^1, Z(\alpha, \underline{r}, \zeta)] = 0$
(2) $[a_0, Z(\alpha, \underline{r}, \zeta)] = \langle a_0, \alpha \rangle Z(\alpha_1, \underline{r}, \zeta), a \in \underline{h}$.

Proof (1) Follows from above. (2) is easy to see.

Proposition (2.3) (Proposition (3.9) of [LW]).

Let W be a vector space and let $f(\zeta_1, \zeta_2) = \sum w_{ij} \zeta_1^i \zeta_2^j$ where each $w_{ij} \in W$ and suppose for some $n \in \mathbb{Z}$ either $w_{ij} = 0$ where one of i or $j > n$ or $w_{ij} = 0$ whenever i or $j < n$.

Set

$$D_i f(\zeta_1, \zeta_2) = \zeta_i \frac{df}{d\zeta_i}(\zeta_1, \zeta_2)$$

Then for $a \neq 0$

$$\begin{aligned}
(1) \quad \delta(a\zeta_1/\zeta_2)f(\zeta_1, \zeta_2) &= \delta(a\zeta_1/\zeta_2)f(\zeta_1, a\zeta_1) \\
&= \delta(a\zeta_1/\zeta_2)f(a^{-1}\zeta_2, \zeta_2) \\
(2) \quad D\delta(a\zeta_1/\zeta_2)f(\zeta_1, \zeta_2) &= (D\delta)(a\zeta_1/\zeta_2)f(\zeta_1, a\zeta_1) + \delta(a\zeta_1/\zeta_2)D_2f(\zeta_1, a\zeta_1) \\
&= D\delta(a\zeta_1/\zeta_2)f(a^{-1}\zeta_2, \zeta_2) - \delta(a\zeta_1/\zeta_2)D_1f(a^{-1}\zeta_2, \zeta_2)
\end{aligned}$$

Proposition (2.4)

$$\begin{aligned}
&\prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{<\theta^p \beta_1, \beta_2>/k} Z(\beta_1, \underline{r}, \zeta_1) Z(\beta_2, \underline{s}, \zeta_2) \\
&- \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_2/\zeta_1)^{<\theta^p \beta_2, \beta_1>/k} Z(\beta_2, \underline{s}, \zeta_2) Z(\beta_1, \underline{r}, \zeta_1) \\
&= E^-(\beta_1, \zeta_1) E^-(\beta_2, \zeta_2) [X(\beta_1, \underline{r}, \zeta_1), X(\beta_2, \underline{s}, \zeta_2)] \\
&\quad E^+(\beta_1, \zeta_1) E^+(\beta_2, \zeta_2)
\end{aligned}$$

Proof Consider

$$\begin{aligned}
&Z(\beta_1, \underline{r}, \zeta_1) Z(\beta_2, \underline{s}, \zeta_2) \\
&= E^-(\beta_1, \underline{r}, \zeta_1) x_{\beta_1}(\underline{r}, \zeta_1) E^+(\beta_1, \underline{r}, \zeta_1) \cdot \\
&\quad E^-(\beta_2, \underline{s}, \zeta_2) x_{\beta_2}(\underline{s}, \zeta_2) E^+(\beta_2, \underline{s}, \zeta_2) \\
&= \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{\frac{<\theta \beta_1, \beta_2>}{k}} \cdot \\
&\quad E^-(\beta_1, \underline{r}, \zeta_1) x_{\beta_1}(\underline{r}, \zeta_1) E^-(\beta_2, \underline{s}, \zeta_2) \\
&\quad E^+(\beta_1, \underline{r}, \zeta_1) x_{\beta_2}(\underline{s}, \zeta_2) E^+(\beta_2, \underline{s}, \zeta_2)
\end{aligned}$$

from 4 (a) of Proposition (2.1).

$$\begin{aligned}
&= \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{\frac{-<\theta \beta_1, \beta_2>}{k}} \cdot \\
&\quad E^-(\beta_1, \underline{r}, \zeta_1) E^-(\beta_2, \underline{s}, \zeta_2) X_{\beta_1}(\underline{r}, \zeta_1) \\
&\quad x_{\beta_2}(\underline{s}, \zeta_2) E^+(\beta_1, \underline{r}, \zeta_1) E^+(\beta_2, \underline{s}, \zeta_2)
\end{aligned}$$

(by 5 (a) and (b) of Proposition (2.1).

Multiplying both sides by the inverse of the first factor on the right, and subtracting the expression obtained by interchanging the roles of the subscripts 1 and 2 we have Proposition (2.4).

Proposition (2.5) For these Z operators the relation at (1.10) hold.

Proof (2) to (6), holds from definition of d_i . Since $V \in \underline{C}_k$ we have

$$\frac{1}{k} X_\beta(\underline{r}, \zeta) k_0(\underline{s}, \zeta^m) = X_\beta(\underline{r} + \underline{s}, \zeta).$$

Thus (1) holds from definition of Z operator. (9) holds from Proposition 1.5 (6), (10) and (8) are easy to see. We only need to prove (7). The RHS of the Proposition (2.4) and by using Proposition 1.5 (1) is equal to

$$E_1 + E_2 + E_3 + E_4$$

$$\begin{aligned} \text{where } E_1 &= E^-(\beta_1, \zeta_1) E^-(\beta_2, \zeta_2). \\ &\quad \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \phi} \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) x_{\theta^p \beta_1 + \beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ &\quad E^+(\beta_1, \zeta_1) E^+(\beta_2, \zeta_2) \\ E_2 &= E^-(\beta_1, \zeta_1) E^-(\beta_2, \zeta_2). \\ &\quad \left(\frac{-1}{m}\right) \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \beta_2(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ &\quad E^+(\beta_1, \zeta_1) E^+(\beta_2, \zeta_2) \\ E_3 &= E^-(\beta_1, \zeta_1) E^-(\beta_2, \zeta_2). \\ &\quad \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \\ &\quad E^+(\beta_1, \zeta_1) E^+(\beta_2, \zeta_2) \\ E_4 &= E^-(\beta_1, \zeta_1) E^-(\beta_2, \zeta_2) \\ &\quad \frac{1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(\underline{r} + \underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1 / \zeta_2) \\ &\quad E^+(\beta_1, \zeta_1) E^+(\beta_2, \zeta_2) \end{aligned}$$

Now

$$E_1 = \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \Phi} \eta(p, \beta_1) E^-(\theta^p \beta_1 + \beta_2, \zeta_2).$$

$$x_{\theta^p \beta_1 + \beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2).$$

$$E^+(\theta^p \beta_1 + \beta_2, \zeta_2).$$

$$E_2 = -\frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \beta_2(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2)$$

$$E_3 = \pm \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2).$$

For E_4 we use Proposition 2.3 (2) ($a = w^{-p}$) and Proposition 2.1 (2) (b).

Thus we get

$$E_4 = \frac{1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(\underline{r} + \underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1 / \zeta_2).$$

$$\frac{-1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \frac{m}{k} \sum_{i \neq 0} (\beta_1)_i \otimes t^i (w^p \zeta_2)^i k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2)$$

We will use the fact that $\sum_{i \neq 0} (\beta_1)_i \otimes t^i \zeta_2^i = \beta_2(\zeta) - (\beta_2)_0$ and the fact that

$$(2.6) \quad \frac{1}{k} \beta_2(\zeta_2) k_0(\underline{r} + \underline{s}, \zeta_2^m) = \beta_2(\underline{r} + \underline{s}, \zeta_2)$$

So we get

$$E_4 = \frac{1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(\underline{r} + \underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1 / \zeta_2)$$

$$-\frac{1}{mk} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \sum_{i \neq 0} (\theta^p \beta_1)_i \otimes t^i \zeta_2^i k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2)$$

$$= \frac{1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(\underline{r} + \underline{s}, \zeta_2^m) D(\delta w^{-p} \zeta_1 / \zeta_2)$$

$$\begin{aligned}
\frac{-1}{mk} \langle x_{\beta_2}, x_{-\beta_2} \rangle &= \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta) \theta^p \beta_1(\zeta_2) k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \\
&+ \frac{1}{mk} \langle x_{\beta_2}, x_{-\beta_2} \rangle = \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) (\beta_1)_0 k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2).
\end{aligned}$$

Note that the second term is $-E_2$ by (2.6). Since $\theta^p \beta_1 + \beta_2 = 0$ we have $(\beta_1)_0 + (\beta_2)_0 = 0$. Thus $E_2 + E_4$

$$\begin{aligned}
&= \frac{1}{m^2} \langle x_{\beta_2}, x_{-\beta_2} \rangle = \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(\underline{r} + \underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1 / \zeta_2) \\
&- \frac{1}{m_k} \langle x_{\beta_2}, x_{-\beta_2} \rangle = \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) (\beta_2)_0 k_0(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2).
\end{aligned}$$

Now adding E_1, E_2, E_3 and E_4 we get the defined result. Thus we proved that $\Omega(V) \in \underline{D}_k$. Conversely assume that $W \in \underline{D}_k$. Let $V = M(k) \otimes W$. Define $X_\alpha(\underline{r}, \zeta) = E^-(-\alpha, \zeta) E^+(-\alpha, \zeta) \otimes Z(\alpha, \underline{r}, \zeta)$.

$$\beta(\underline{r}, \zeta) = \frac{1}{k} \beta(z) k_0(\underline{r}, \zeta^m).$$

The central elements to be same. Since $W \in \Omega_k$. The operators $X_\alpha(\underline{r}, \zeta)$ and $k_0(\underline{r}, \zeta)$ satisfy

$$\frac{1}{k} X_\alpha(\underline{r}, \zeta) k_0(\underline{s}, \zeta^m) = X(\underline{r} + \underline{s}, \zeta) \text{ for all } X_\alpha \in \mathcal{G}.$$

Conditions 4 to 7 of Proposition (1.6) are easily satisfied as the corresponding conditions are satisfied for Z -operators.

Condition (3) can be proved exactly as in the proof of Proposition 5.3 of [LW]. Condition (1) is satisfied as the same relation holds for Z -operators and Z operator commutes with E^\pm -operators. Consider $[\beta_1(\zeta_1), \beta_2(\zeta_2)] = \frac{1}{m^2} k_0 \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle D \delta(w^{-p} \zeta_1 / \zeta_2)$. See Theorem 2.4 of [LW].

$$\begin{aligned}
[\beta_1(\underline{r}, \zeta_1), \beta_2(\underline{s}, \zeta_2)] &= [\beta_1(\zeta_1), \beta_2(\zeta_2)] k_0(\underline{r}, \zeta_1^m) k_0(\underline{s}, \zeta_2^m) \\
&= \frac{1}{km^2} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle k_0(\underline{r}, \zeta_1^m) k_0(\underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1 / \zeta_2)
\end{aligned}$$

Now by Proposition 2.3 (2) we have

$$\begin{aligned}
&= \frac{1}{m^2} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle k_0(\underline{r} + \underline{s}, \zeta_1^m) D\delta(w^{-p}\zeta_1/\zeta_2) \\
&\quad - \frac{1}{mk} \sum \langle \theta^p \beta_1, \beta_2 \rangle D_1 k_0(\underline{r}, \zeta_1^m) |_{\zeta_1=w^p\zeta_2} k_0(\underline{s}, \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2) \\
&= \frac{1}{m^2} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle k_0(r + s, \zeta_2^m) D\delta(w^{-p}\zeta_1/\zeta_2) \\
&\quad + \frac{1}{m} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle \sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2) \\
&\quad \quad \quad \sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2)
\end{aligned}$$

Thus we have proved the following:

Proposition 2.6 The category $\underline{\mathcal{C}}_k$ of $\tilde{L}(\mathcal{G}, \theta)$ -modules are equivalent to the category $\underline{\mathcal{D}}_k$ of \mathbb{Z}_k -modules.

Section 3 (Homogeneous picture)

In this section our aim is to construct a faithful representation for the untwisted toroidal Lie algebra $\tilde{\tau}$ coming from simple, simply connected Lie-algebra \mathcal{G} . (First note that on any representation τ where centre Ω_A/d_A acts faithfully, then $\tilde{\tau}$ acts faithfully). That is we are giving a realization. This recovers the main result of [EM]. For this we give a representation for the Z -algebra such that the centre acts faithfully. Thus we have a faithful representation for the toroidal Lie algebra τ .

We take the automorphism $\theta = id$. We first give a presentation for the Lie-algebra \mathcal{G} . Let $\overset{\circ}{Q}$ be the root lattice spanned by simple roots. The

nondegenerate form is chosen so that $(\alpha, \alpha) = 2$ for a highest root α . Then it is known that

$$\phi = \{\alpha \in \overset{\circ}{Q} \mid (\alpha, \alpha) = 2.\}$$

The following cocycle on $\overset{\circ}{Q} \times \overset{\circ}{Q}$ is known to exist.

$$\epsilon : \overset{\circ}{Q} \times \overset{\circ}{Q} \rightarrow \{\pm 1\}$$

3.1

$$(1) \quad \epsilon(\alpha, \alpha) = (-1)^{\frac{(\alpha, \alpha)}{2}}$$

$$(2) \quad \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$$

$$(3) \quad \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$$

$$(4) \quad \epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$$

Note that $\epsilon(\alpha, \alpha) = \epsilon(\alpha, -\alpha) = -1$ for $\alpha \in \phi$. Then there exist vectors x_α, h_α in \mathcal{G} $\alpha \in \phi$ satisfying the following:

(3.2)

$$\begin{aligned} (1) \quad [x_\alpha, x_\beta] &= \epsilon(\alpha, \beta)x_{\alpha+\beta}, \alpha + \beta \in \phi \\ (2) \quad [x_{\alpha_1}x_{-\alpha}] &= 0 \text{ if } \alpha + \beta \notin \phi \cup \{0\} \\ &= \epsilon(\alpha_1, -\alpha_1)h_\alpha \text{ if } \alpha + \beta = 0 \\ (3) \quad [h_\alpha, h_\beta] &= 0 \\ (4) \quad [h, x_\alpha] &= \alpha(h)x_\alpha \quad \forall h \in \underline{h}. \end{aligned}$$

Let Γ be a \mathbb{Z} -lattice spanned by $\alpha_1, \dots, \alpha_\ell, \delta_1, \dots, \delta_N, d_1, \dots, d_N$. Define a non-degenerate bilinear form on Γ extending the one on $\overset{\circ}{Q}$ by

$$\begin{aligned} (\overset{\circ}{Q}, \delta_i) &= (\overset{\circ}{Q}, d_i) = 0 \\ (d_i, d_j) &= (\delta_i, \delta_j) = 0 \\ (\delta_i, d_j) &= \delta_{ij} \end{aligned}$$

Any vector which is integral linear combination of δ_i is called a null root. For $\underline{r} \in \mathbb{Z}^N$ define $\delta_{\underline{r}} = \sum r_i \delta_i$. Note that $(\delta_{\underline{r}}, \delta_{\underline{s}}) = 0$. Let Q be the sub lattice spanned by \vec{Q} and $\delta_1, \dots, \delta_n$. Extend the co-cycle ϵ to Q by $\epsilon(\alpha, \delta_{\underline{r}}) = 1$ for $\alpha \in Q$. Now extend ϵ to $Q \times \Gamma$ to be bimultiplicative in any convenient way. Consider the group algebra $\mathbb{C}[\Gamma]$ and make $\mathbb{C}[\Gamma]$ a $\mathbb{C}[Q]$ module by the following multiplication.

$$e^\alpha \cdot e^\gamma = \epsilon(\alpha, \gamma) e^{\alpha+\gamma}.$$

Let $\bar{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\bar{h}_\pm = \bigoplus_{n \gtrless 0} \bar{h} \otimes t^n$. Consider the Fock space

$$V(\Gamma) = S(\bar{h}) \otimes \mathbb{C}[\Gamma].$$

Define operators $\alpha(0)$ on $V(\Gamma)$ by

$$\alpha(0) \cdot u \otimes e^\gamma = (\alpha, \gamma) u \otimes e^\gamma, \alpha \in Q.$$

For δ nullroot

$$\delta(n) u \otimes e^\gamma = \delta(n) u \otimes e^\gamma, n \neq 0$$

$\delta(n) u$ is multiplication if $n < 0$ and differentiation if $n > 0$. This is the standard Fock space representation of $\bar{h} \otimes \mathbb{C}[t, t^{-1}]$ on $V[\Gamma]$.

For a null root δ define $E^\pm(\delta, \zeta) = \exp \sum_{n \gtrless 0} \frac{\alpha(\pm n)}{\pm n} \zeta^{\pm n}$. Define operators

$$\zeta^{\alpha(0)} u \otimes e^\gamma = \zeta^{(\alpha, \gamma)} u \otimes e^\gamma, \alpha \in Q.$$

Consider the vertex operator

$$X(\delta, \zeta) = E^-(\delta, \zeta) \zeta^{\delta(0)} E^+(\delta, \zeta).$$

Let $k_i(\delta, \zeta) = \delta_i(\zeta) X(\delta, \zeta)$ for $1 \leq i \leq N$ and $k_0(\delta, \zeta) = X(\delta, \zeta)$. From [EM] it is known that each $k_i(\delta, \zeta)$ acts non trivially and $Dk_0(\delta_{\underline{\gamma}}, \zeta) + \sum_{i=1}^N r_i k_i(\delta_{\underline{\gamma}}, \zeta) =$

0. Further any relation among $k_i(\delta, \zeta)$ is the one given above. (see Lemma C of [EM]).

We will now define Z operators. Define $Z(\alpha, 0, \zeta) = \zeta^{\frac{(\alpha, \alpha)}{2}} \zeta^{-\alpha} e^\alpha, \alpha \in \phi$. Then define $Z(\alpha, \underline{r}, \zeta) = Z(\alpha, 0, \zeta) k_0(\underline{r}, \zeta)$. d_0, d_1, \dots, d_N are defined naturally as grading on $Z(\alpha, \underline{r}, \zeta)$.

We will now check the relation at (1.10) for the above Z -operator. (1) to (6) are clearly satisfied from definition. We will rewrite the relation (7) using the fact that $\theta = Id$ and $m = 1$.

(3.3)

$$\begin{aligned}
& (1 - \zeta_1/\zeta_2)^{<\beta_1, \beta_2>} Z(\beta_1, \underline{r}, \zeta_1) Z(\beta_2, \underline{s}, \zeta_2) - (1 - \zeta_2/\zeta_1)^{<\beta_1, \beta_2>} Z(\beta_2, \underline{s}, \zeta_2) Z(\beta_1, \underline{r}, \zeta_1) \\
&= \epsilon(\beta_1, \beta_2) Z(\beta_1 + \beta_2, \underline{r} + \underline{s}, \zeta_2) \delta(\zeta_1/\zeta_2) \text{ if } \beta_1 + \beta_2 \in \Phi \\
&= <x_{\beta_2}, x_{-\beta_2}> \beta_2 \frac{1}{k} k_0(\underline{r} + \underline{s}, \zeta_2) \delta(\zeta_1/\zeta_2) \\
&+ <x_{\beta_2}, x_{-\beta_2}> \left(\sum_{i=1}^N r_i k_i(\underline{r} + \underline{s}, \zeta_2) \cdot \delta(\zeta_1/\zeta_2) + k_0(\underline{r} + \underline{s}, \zeta_2) D \delta(\zeta_1/\zeta_2) \right) \text{ if } \beta_1 + \beta_2 = 0 \\
&= 0 \text{ if } \beta_1 + \beta_2 \notin \Phi \cup \{0\}.
\end{aligned}$$

Suppose $\underline{r} = 0$ and $\underline{s} = 0$ then (3.3) follows from Theorem 5.3 of [LM]. For general \underline{r} and \underline{s} , consider L.H.S of (3.3) which is equal to

$$\begin{aligned}
& (1 - \zeta_1/\zeta_2)^{<\beta_1, \beta_2>} Z(\beta_1, 0, \zeta_1) k_0(\underline{r}, \zeta_1) Z(\beta_2, 0, \zeta_2) k_0(\underline{s}, \zeta_2) \\
& - (1 - \zeta_1/\zeta_2)^{<\beta_1, \beta_2>} Z(\beta_2, 0, \zeta_2) k_0(\underline{s}, \zeta_2) \cdot Z(\beta_1, 0, \zeta_1) k_0(\underline{r}, \zeta_1) \\
&= k_0(\underline{r}, \zeta_1) k_0(\underline{s}, \zeta_2) Z(\beta_1, \beta_2) \text{ where } Z(\beta_1, \beta_2) \\
&= \epsilon(\beta_1, \beta_2) Z(\beta_1 + \beta_2, 0, \zeta_1) \delta(\zeta_1/\zeta_2) \text{ if } \beta_1 + \beta_2 \in \Phi \\
&+ <x_{\beta_2}, x_{-\beta_2}> (-\beta_2 \delta(\zeta_1/\zeta_2) + k D \delta(\zeta_1/\zeta_2)) \text{ if } \beta_1 + \beta_2 = 0 \\
&= 0 \text{ if } \beta_1 + \beta_2 \notin \Phi \cup \{0\}.
\end{aligned}$$

This follows from case $\underline{r} = 0 = s$. Now the case $\beta_1 + \beta_2 \in \Phi$, (3.3) follows from Proposition 2.3 (1). The case $\beta_1 + \beta_2 \notin \Phi \cup \{0\}$ is trivial. For the case $\beta_1 + \beta_2 = 0, \langle x_{\beta_2}, x_{-\beta_2} \rangle k_0(\underline{r}, \zeta_1)k_0(\underline{r}, \zeta_2)\beta_2\delta(\zeta_1/\zeta_2)$ is equal to the first term of 3.3 which follows from Proposition 2.3(1).

Now

$$\begin{aligned} & \langle x_{\beta_2}, x_{-\beta_2} \rangle k_0(\underline{r}, \zeta_1)k_0(\underline{s}, \zeta_2).kD\delta(\zeta_1/\zeta_2) \\ & = \langle x_{\beta_2}, x_{-\beta_2} \rangle (k_0(\underline{r} + \underline{s}, \zeta_2)D\delta(\zeta_1/\zeta_2) + \sum r_i k_i(\underline{r} + \underline{s}, \zeta_2)\delta(\zeta_1/\zeta_2)). \end{aligned}$$

This completes the proof of (3.3).

Section 4 Principal realization

Let \mathcal{G} be a simple finite dimensional Lie-algebra. Let π be an automorphism of order $K(= 1, 2 \text{ or } 3)$ induced by an automorphism of the Dynkin diagram of \mathcal{G} with respect to some Cartan subalgebras h of \mathcal{G} . Let ϵ be K -th primitive root. We will now extend the automorphism π to $\mathcal{G} \otimes A \oplus \Omega_A/d_A = \tau$ by

(4.1)

$$\begin{aligned} \pi(xt^{r_0}t^{\underline{r}}) &= \epsilon^{-r_0}\pi(x)t^{r_0}t^{\underline{r}} \\ \pi(t^{r_0}t^{\underline{r}}) &= \epsilon^{-r_0}t^{r_0}t^{\underline{r}}, \quad 0 \leq i \leq N. \end{aligned}$$

The aim of this section is to prove that $\overline{L}(\mathcal{G}, \pi) \cong \overline{L}(\mathcal{G}, \theta)$ where θ is a special automorphism depending on π . This is a generalisation of the standard principal realization of affine Lie-algebras given in [KKLW].

To do this we first have to define the automorphism θ . For $i \in \mathbb{Z}$, let $\mathcal{G}_{[i]}$ be the ϵ^i eigenspace of \mathcal{G} . Then the fixed point space $\mathcal{G}_{[0]}$ is a simple Lie-subalgebra of \mathcal{G} and $\mathcal{G}_{[0]}$ module $\mathcal{G}_{[1]}$ and $\mathcal{G}_{[-1]}$ are irreducible and contra-gradient.

Fix a Cartan subalgebra \underline{t} of $\mathcal{G}_{[0]}$ inside \underline{h} . Let H_j, E_j, F_j ($1 \leq j \leq \ell$) be a corresponding set of canonical generators of $\mathcal{G}_{[0]}$. Let E_0 be the lowest

weight vector of $\mathcal{G}_{[0]}$ module $\mathcal{G}_{[1]}$, and let F_0 be the highest weight vector, $\mathcal{G}_{[0]}$ -module $\mathcal{G}_{[-1]}$, normalised so that $[H_0, F_0] = 2F_0$ where $H_0 = [E_0, F_0]$. Let $\psi_1, \dots, \psi_{\ell} \in \underline{t}^*$ be simple roots of $\mathcal{G}_{[0]}$, and let $\psi_0 \in \underline{t}^*$ be the lowest weight of the $\mathcal{G}_{[0]}$ module $\mathcal{G}_{[1]}$. For $i, j = 0, 1, \dots, \ell$ set $A_{ij} = \psi_j(H_i)$. Then it is known that $A = (A_{ij})$ is an indecomposable affine Cartan matrix (see [LW] and [KKLW]). Let $a_0, \dots, a_\ell, a_0^1, \dots, a_\ell^1$ be positive integers such that

(4.2)

$$\begin{aligned} A(a_0, \dots, a_\ell)^T &= 0 \\ (a_0^1, \dots, a_\ell^1)A &= 0 \text{ and} \\ g.c.d(a_0, \dots, a_\ell) &= 1 \\ g.c.d(a_0^1, \dots, a_\ell^1) &= 1. \end{aligned}$$

Then a_0, a_1, \dots, a_ℓ are precisely the indices of the Dynkin diagram of A . (see Table K of [KKLW]).

(4.3)

Note that from above tables we see that $a_0 = 1$ always. Recall from [LW] that

(4.4)

$$\begin{aligned} \sum_{j=0}^{\ell} a_j \psi_j &= 0 \text{ and} \\ \sum_{j=0}^{\ell} a_j^1 H_j &= 0 \end{aligned}$$

(4.5) Proposition (KKLW) The Lie subalgebra G of $\mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$ generated by $E_i, F_i, H_i (1 \leq i \leq \ell), E_0 \otimes t, F_0 \otimes t^{-1}, H_0$ and C is isomorphic to the affine Lie algebra corresponding to A .

Let $\underline{s} = (s_0, \dots, s_\ell)$ be a sequence of non-negative integers, not all 0.

$$\text{Take } m = K \sum_{j=0}^{\ell} s_j a_j.$$

Define an automorphism θ of \mathcal{G} by the condition

$$\theta H_i = H_i, \theta E_j = w^{s_j} E_j.$$

Then θ defines an automorphism of \mathcal{G} of order m . See [LW]. Then from section 1 we can define $\overline{L}(\mathcal{G}, \theta)$. Our aim in this section is to prove $\overline{L}(\mathcal{G}, \pi) \cong \overline{L}(\mathcal{G}, \theta)$ which is what we call principal realization of toroidal Lie algebras.

(4.6) As earlier $<, >$ be a non-degenerate \mathcal{G} -invariant bilinear form on \mathcal{G} which is necessarily θ and π invariant. This form $<, >$ remains non-singular on the Cartan subalgebra \underline{t} of $\mathcal{G}_{[0]}$. See section 8 of [LW].

Using the restricted form we identify \underline{t} and \underline{t}^* . We normalise the form $<, >$ such that

$$(4.7) \quad < \psi_0, \psi_0 > = \frac{2a_o^1}{K}.$$

Then we have $a_j^1 = K < \psi_j, \psi_j > a_j/2$ for $j = 0, \dots, \ell$. (see [LW]).

Now we have the following \underline{s} -realization of the affine Lie algebra G from [KKLW].

(4.8) **Proposition** ([LW], [KKLW])

Let $e_j = E_j \otimes t^{s_j}, f_j = F_j \otimes t^{-s_j}, h_j = H_j \otimes 1 + 2s_j < \psi_j, \psi_j >^{-1} m^{-1} C$ inside $\widehat{\mathcal{G}}$. Then there is an isomorphism of affine Lie-algebras $\varphi : G \rightarrow \widehat{\mathcal{G}}$

defined by

$$\begin{aligned}
\phi(E_i \otimes 1) &= e_i \quad 1 \leq i \leq \ell \\
\varphi(F_i \otimes 1) &= f_i, 1 \leq i \leq \ell \\
\varphi(E_0 \otimes t) &= e_0 \\
\varphi(F_0 \otimes t^{-1}) &= f_0 \\
\varphi(H_i) &= h_i \quad 1 \leq i \leq \ell \\
\varphi(H_0 + \frac{\langle E_0, F_0 \rangle}{K} C) &= h_0.
\end{aligned}$$

Further e_i, f_i, h_i ($0 \leq i \leq \ell$) forms a set of canonical generators for the affine Lie-algebra $\widehat{\mathcal{G}}$. Here the Lie bracket G is defined by the bilinear form $\frac{1}{K} \langle \cdot, \cdot \rangle$.

As we are interested in the principal realization we take $\underline{s} = (1, 1, \dots, 1)$.

(4.9) Remark. The π -invariants of \underline{h} equal to \underline{t} . In particular they are spanned by $H_i, 1 \leq i \leq \ell$.

(4.10) Proposition Let w be the Chevalley involution automorphism of \mathcal{G} . Let $\phi : G \rightarrow \widehat{\mathcal{G}}$ be the isomorphism of Lie-algebras given earlier. Then the following hold.

(1) $\varphi(C) = C$

(2) $\varphi(x_\alpha \otimes t^{r_0}) = x_\alpha \otimes t^{N(\alpha) + \frac{m}{k} r_0}$ where $x_\alpha \otimes t^{r_0}$ is a real root vector of G .
 $N(\alpha)$ is an integer independent of r_0 but depends on α .

(3) Let $h_\alpha = [x_\alpha, w(x_\alpha)]$ then

$$\varphi(h_\alpha \otimes t^{r_0}) = h_\alpha \otimes t^{\frac{m}{k} r_0} + \langle x_\alpha, w(x_\alpha) \rangle \frac{N_\alpha}{m} \delta_{r_0, 0}$$

(4) $\langle x_\alpha, x_\beta \rangle \neq 0$ implies $N(\alpha) + N(\beta) = 0$

(5) $\langle x_\alpha, x_\beta \rangle = 0$ then

$$\varphi([x_\alpha, x_\beta] \otimes t^{r_0 + s_0}) = [x_\alpha, x_\beta] \otimes t^{(r_0 + s_0) \frac{m}{k} + N(\alpha) + N(\beta)}.$$

Here $[x_\alpha, x_\beta]$ could be part of real or imaginary root.

Proof Let $x_\alpha \otimes t^{r_0}$ be a real root vector of G . Then G is spanned by $x_\alpha \otimes t^{r_0}, [x_\alpha, x_\beta] \otimes t^{r_0}$ and C . We have from (4.4)

$$H_0 = -\sum_{j=1}^{\ell} \frac{a_j^1}{a_o^1} H_j.$$

From Proposition (4.8) we have

$$H_0 \otimes 1 + 2 \langle \psi_0, \psi_0 \rangle^{-1} m^{-1} C = h_0 = [E_0 t, F_0 t^{-1}] = [E_0, F_0] + \langle E_0, F_0 \rangle \frac{1}{m} C.$$

(4.11) This implies $\langle E_0, F_0 \rangle = \frac{2}{\langle \psi_0, \psi_0 \rangle}$. Consider

$$\begin{aligned} \phi(H_0) &= -\sum_{j=1}^{\ell} \frac{a_j^1}{a_o^1} \phi(H_j) \\ &= -\sum_{j=1}^{\ell} \frac{a_j^1}{a_o^1} (H_j + \frac{2}{\langle \psi_j, \psi_j \rangle} \frac{1}{m} C) \\ &= H_0 - \sum_{j=1}^{\ell} \frac{a_j^1}{a_o^1} \frac{1}{m} \frac{a_j}{a_j^1} K C \\ &= H_0 - \sum_{j=1}^{\ell} \frac{a_j K C}{a_o^1 m} \\ &= H_0 - \frac{(m - a_0 k)}{a_o^1 m} C \\ \text{But } (H_0 + \frac{\langle E_0, F_0 \rangle}{K} C) &= H_0 + \frac{2}{\langle \psi_0, \psi_0 \rangle} \frac{1}{m} C \end{aligned}$$

(by Proposition 4.8)

$$\begin{aligned} \varphi(C) \frac{2}{K \langle \psi_0, \psi_0 \rangle} &= H_0 + \frac{2}{\langle \psi_0, \psi_0 \rangle} \frac{1}{m} C - \varphi(H_0) \\ &= \frac{2}{(\psi_0, \psi_0)} \frac{1}{m} C + \frac{1}{a_o^1} C - \frac{a_0 K}{a_o^1 m} C \\ \frac{\varphi(C)}{a_o^1} &= \left(\frac{K}{a_o^1} \frac{1}{m} + \frac{1}{a_o^1} - \frac{a_0 K}{a_o^1 m} \right) C \end{aligned}$$

(We are using (4.10), (4.7). This implies $\varphi(C) = C$ by (4.3).

(2) Clearly $\varphi(x_\alpha \otimes t^{r_0}) = x_\alpha \otimes t^N$ for some integer. Write $N = N_\alpha + r_0 \frac{m}{K}$ for some integer N_α which may depend on r_0 . Clearly $w(x_\alpha) t^{-r_0} = w(x_\alpha) \otimes$

$t^{-r_0 \frac{m}{k} - N_\alpha}$. Let $h_\alpha = [x_\alpha, w(x_\alpha)]$. Consider the following in G .

$$[x_\alpha \otimes t^{r_0}, w(x_\alpha) \otimes t^{-r_0}] = h_\alpha + \frac{\langle x_\alpha, w(x_\alpha) \rangle r_0 C}{K}$$

Now apply φ both sides and the bracket takes place in $\widehat{\mathcal{G}}$

$$\varphi(h_\alpha + \frac{r_0}{K} \langle x_\alpha, w(x_\alpha) \rangle) = h_\alpha + \frac{\langle x_\alpha, w(x_\alpha) \rangle}{m} (N_\alpha + \frac{m}{K} r_0) C).$$

Since $\varphi(C) = C$ we have

$$\varphi(h_\alpha) = h_\alpha + \langle x_\alpha, w(x_\alpha) \rangle \frac{N_\alpha}{m} C.$$

As h_α is independent of r_0 it follows that N_α does not depends on r_0 . Now consider the following in G .

$$[x_\alpha \otimes t^{r_0}, w(x_\alpha) \otimes t^{s_0}] = h_\alpha \otimes t^{r_0+s_0} + (x_\alpha, w(x_\alpha) r_0 \delta_{r_0+s_0,0} C).$$

As earlier apply φ both sides

$$\varphi(h_\alpha \otimes t^{r_0+s_0}) = h_\alpha \otimes t^{(r_0+s_0) \frac{m}{k}} + \langle x_\alpha, w(x_\alpha) \rangle \frac{N_\alpha}{m} \delta_{r_0+s_0,0} C.$$

This proves (3). For (4) suppose $\langle x_\alpha, x_\beta \rangle \neq 0$. Since \langle, \rangle is \underline{t} -invariant, it follows that $\alpha + \beta$ is zero root. (root with respect to \underline{t}). Thus $[x_\alpha, x_\beta]$ is a part of imaginary root and so $[x_\alpha, x_\beta] \in \underline{h}$. Since π is an automorphism we have

$$\langle \pi(x_\alpha), \pi(x_\beta) \rangle = \langle x_\alpha, x_\beta \rangle \neq 0.$$

Let $\pi(x_\alpha) = e^i x_\alpha$ and $\pi(x_\beta) = e^j x_\beta$. Since $\langle x_\alpha, x_\beta \rangle \neq 0$ it follows that $i + j \equiv 0(K)$. Thus $[x_\alpha, x_\beta]$ is π -invariant. Consider

$$\begin{aligned} [x_\alpha t^{r_0}, x_\beta t^{s_0}] &= [x_\alpha, x_\beta] t^{r_0+s_0} + \langle x_\alpha, x_\beta \rangle r_0 \delta_{r_0+s_0,0} C \\ \varphi([x_\alpha, x_\beta] t^{r_0+s_0}) &= [x_\alpha, x_\beta] t^{(r_0+s_0) \frac{m}{k} + N_{(\alpha)} + N_{(\beta)}} \\ &+ \langle x_\alpha, x_\beta \rangle N C \text{ for some } N. \end{aligned}$$

Now from (3) and (4.9) it follows that

$$\varphi[x_\alpha, x_\beta]t^{r_0+s_0} = [x_\alpha, x_\beta]t^{(r_0+s_0)}\frac{m}{k}.$$

This forces $N_\alpha + N_\beta = 0$.

(5) is clear.

Now we define an isomorphism between $\overline{L}(\mathcal{G}, \pi)$ and $\overline{L}(\mathcal{G}, \theta)$.

(4.12) Proposition The following map φ define an isomorphism from $\overline{L}(\mathcal{G}, \pi)$ to $\overline{L}(\mathcal{G}, \theta)$.

$$\begin{aligned} (1) \quad \varphi(x_\alpha \otimes t^{r_0}t^x) &= x_\alpha \otimes t^{r_0\frac{m}{k}+N_\alpha}t^x \\ (2) \quad \varphi(h_\alpha t^{r_0}t^x) &= h_\alpha t^{r_0\frac{m}{k}+N_\alpha}t^x \\ &\quad + \langle x_\alpha, w(x_\alpha) \rangle \frac{N_\alpha}{m}t^{\frac{r_0m}{k}}t^x k_0 \\ (3) \quad \varphi(t^{r_0}t^x k_i) &= t^{r_0\frac{m}{k}}t^x k_i, 0 \leq i \leq N \\ (4) \quad \text{if } \langle x_\alpha, x_\beta \rangle &= 0 \\ \varphi([x_\alpha, x_\beta]t^{r_0}t^x) &= [x_\alpha, x_\beta]t^{r_0\frac{m}{k}+N(\alpha)+N(\beta)}t^x. \end{aligned}$$

Proof In view of earlier Proposition the RHS belongs to $\overline{L}(\mathcal{G}, \theta)$ except possible for (3). For (3) $t^{r_0}t^x k_i \in \overline{L}(\mathcal{G}, \pi)$ which means $r_0 \equiv (K)$. Thus $r_0\frac{m}{k} \equiv 0(m)$ which means $t^{r_0\frac{m}{k}}t^x k_i$ belongs $\overline{L}(\mathcal{G}, \theta)$. The fact that φ defines an isomorphism follows by the corresponding isomorphism of the earlier proposition. We will verify one bracket. Consider

$$\begin{aligned} \mathbf{4.13} \quad [x_\alpha t^{r_0}t^x, x_\beta t^{s_0}t^s] &= [x_\alpha, x_\beta]t^{r_0+s_0}t^{x+s} \\ &\quad + \frac{\langle x_\alpha, x_\beta \rangle}{K}t^{r_0+s_0}t^{x+s}k_0 + \langle x_\alpha, x_\beta \rangle \sum r_i t^{r_0+s_0}t^{x+s}k_i \end{aligned}$$

Suppose $\langle x_\alpha, x_\beta \rangle = 0$. Then the φ of both sides are equal. Suppose $\langle x_\alpha, x_\beta \rangle \neq 0$, then by previous proposition it follows that $[x_\alpha, x_\beta]$ is in \underline{h}

and π -invariant. Further $N(\alpha) + N(\beta) = 0$.

$$\begin{aligned}
[\varphi(x_\alpha t^{r_0} t^x), \varphi(x_\beta t^{s_0} t^s)] &= [x_\alpha t^{r_0 \frac{m}{k} + N_\alpha} t^x, x_\beta t^{s_0 \frac{m}{k} + N_\beta} t^s] \\
&= [x_\alpha, x_\beta] t^{(r_0 + s_0) \frac{m}{k}} t^{x+s} \\
&\quad + \frac{\langle x_\alpha, x_\beta \rangle}{m} (N_\alpha + \frac{r_0 m}{k}) t^{(r_0 + s_0) \frac{m}{k}} t^{r+s} k_0 \\
&\quad + \langle x_\alpha, x_\beta \rangle \sum r_i t^{(r_0 + s_0) \frac{r}{k}} t^{r+s} k_i.
\end{aligned}$$

which is exactly equal to φ of the RHS of (4.13).

The remaining brackets are easily verified. In this section while defining Lie-bracket structure on $\overline{L}(\mathcal{G}, \pi)$ we have taken the following lie bracket.

$$\begin{aligned}
[xt^{r_0} t^x, yt^{r_0} t^s] &= [x, y] t^{r_0 + s_0} t^{r+s} \\
&\quad + \frac{(x, y)}{K} t^{r_0 + s_0} t^{r+s} k_0 \\
&\quad + (x, y) \sum r_i t^{r_0 + s_0} t^{r+s} k_i
\end{aligned}$$

and Ω_A/d_A is span by $t^{r_0} t^x k_i$ with relation $\sum r_i t^{r_0} t^x k_i + \frac{1}{K} r_0 t^r t^x k_0 = 0$. This algebra is clearly isomorphic to $\overline{L}(\mathcal{G}, \pi)$ that has been defined in section where we took $K = 1$. Thus $\overline{L}(\mathcal{G}, \pi)$ is isomorphic to $\overline{L}(\mathcal{G}, \theta)$ without the denominator K .

(4.14) Proposition $\overline{L}(\mathcal{G}, \pi)$ is the universal central extension of $L(\mathcal{G}, \pi)$. Follows from Remark (2.4) of [BK].

In the next section we give a faithful realisation to $\overline{L}(\mathcal{G}, \theta)$ thereby giving a realization to $\overline{L}(\mathcal{G}, \pi)$ where the infinite dimension centre acts faithfully.

Section 5 Principal picture.

In this section we construct level one module for the toroidal Lie-algebra of type A_n^K, D_n^K and E_n^K . What we do is to construct representation for the Z algebras where the centre acts faithfully. That in turn constructs module for toroidal algebras of type ADE. This also covers the twisted case which is new result.

Notation as in section 4. Consider the cyclic element $E = \sum_{i=1}^{\ell} E_i \epsilon \mathcal{G}_{(1)}$. We make the assumption that the θ -stable Cartan subalgebra \underline{t} is the centraliser of E . (See [LW] for details).

Let $t_0 = t \oplus \mathbb{C}c \oplus \mathbb{C}d$. Let $L(\lambda)$ be a basic module for $\tilde{L}(\mathcal{G}, \theta)$ where $\lambda \in t_0^*$. We renormalize the root vector $x_\beta, \beta \in \phi$ such that $[x_\beta, x_{-\beta}] = -2/ \langle \beta, \beta \rangle$ and $\eta(p, \beta) = 1$ for all $p \in \mathbb{Z}_m$ and for all $\beta \in \phi$. As in theorem 8.7 of [LW] choose coset representatives $\beta_1, \dots, \beta_\ell$ for the action θ on ϕ such that $(\beta_1)_0, \dots, (\beta_\ell)_0$ is a basis for \underline{t} . Let $C_j = \lambda(x_{\beta_j})_0$.

Proposition (5.1) We have the following from section (8) of [LW].

$$(1) \dim \Omega_{L(\lambda)} = 1$$

$$(2) Z(\beta_j, \zeta) = C_j \text{ on } \Omega_{L(\lambda)}$$

$$(3) Z(\theta^p \beta_j, \zeta) = Z(\beta_j, w^p \zeta)$$

$$(4) \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{\langle \theta^p \beta_1, \beta_2 \rangle} Z(\beta_1, \zeta_1) Z(\beta_2, \zeta_2)$$

$$\begin{aligned} & - \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_2 / \zeta_1)^{\langle \theta^p \beta_1, \beta_2 \rangle} Z(\beta_2, \zeta_2) Z(\beta_1, \zeta_1) \\ &= \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \phi} \epsilon(\theta^p \beta_1, \beta_2) Z(\theta^p \beta_1 + \beta_2, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2) \\ & \quad - 2m^{-2} k \langle \beta_1, \beta_1 \rangle^{-1} \sum_{\theta^p \beta_1 + \beta_2 = 0} D \delta(w^{-p} \zeta_1 / \zeta_2). \end{aligned}$$

Proof (1), (2) follows from section 8 of [LW]. For that just note that $\eta(p, \beta) = 1$. (4) follows from Theorem 8.7 of [LW] as Z operator defined in (2) satisfy (8.21) of [LW]. We are also using the fact that $\underline{a}_0 = 0$. Γ be \mathbb{Z} -lattice spanned by $\delta_1, \dots, \delta_N, d_1, \dots, d_n$. with bilinear form $(\delta_i, d_j) = \delta_{ij}$

and $(\delta_i, \delta_j) = (d_i, d_j) = 0$. Let $H = \oplus \mathbb{C}\delta_i$ and $H_+ = \bigoplus_{n \in \mathbb{Z}_+, i} \delta_i(n)$. Consider the symmetric algebra $S(H_+)$.

Consider the space $V(\Gamma) = S(H_+) \otimes e^\Gamma$ where e^Γ group algebra.

Let

$$\begin{aligned} E^+(\delta, \zeta^m) &= \exp \sum_{n>0} \frac{\alpha(n)}{n} \zeta^{mn} \\ E^-(\delta, \zeta^m) &= \exp \sum_{n>0} \frac{\alpha(-n)}{-n} \zeta^{-mn} \end{aligned}$$

which act on $S(H_+)$ and on $V(\Gamma)$.

let $\delta(0)$ act on $V(\Gamma)$ by

$$\begin{aligned} \delta(0)u \otimes e^r &= (\delta, r)u \otimes e^r \\ \zeta^{\delta(0)}u \otimes e^r &= \zeta^{(\delta, r)}u \otimes e^r \\ Cu \otimes e^r &= 1u \otimes e^r \end{aligned}$$

Consider $X(\delta, \zeta^m) = E^-(\delta, \zeta^m) z^{m\delta(0)} E^+(\delta, \zeta^m)$

Let $\underline{r} \in \mathbb{Z}^N$ and let $\delta_{\underline{r}} = \sum r_i \delta_i$

Let $K_0(\underline{r}, \zeta^m) = X(\delta_{\underline{r}}, \zeta^m)$

Let $\delta(\zeta^m) = \sum \zeta^{mn}$

Let $k_i(\underline{r}, \zeta^m) = \delta_i(\zeta^m) X(\delta_{\underline{r}}, \zeta^m)$. Then by standard argument one can prove that

$$(5.2) \quad Dk_0(\underline{r}, \zeta^m) = -m \sum_{i=1}^N r_i k_i(\underline{r}, \zeta^m).$$

Now define $Z(\alpha, \underline{r}, \zeta) = Z(\alpha, \zeta) k_0(\underline{r}, \zeta)$. Now we will check all the relation define at (1.10). Remember $k = 1$. (1) is true by definition. (2) is true by the fact that $X(\delta, \zeta^m) X(\delta_2, \zeta^m) = X(\delta_1 + \delta_2, \zeta^m)$ (3) is just (5.2) (4), (5), (6) can be easily checked. (8) is clear, (9) is true as $\eta(p, \beta) = 1$. (10) is true by definition. Thus it remains to prove (7). To see this multiply 4 of Prop 5.1 by $k_0(\underline{r}, \zeta^m) k_0(\underline{s}, \zeta_2^m)$. Consider $k_0(\underline{s}, \zeta_1^m) k_0(\underline{s}, \zeta_2^m) D \delta(w^{-p} \zeta_1, \zeta_2)$ which is equal to (from Proposition 2.3(2))

$$\begin{aligned}
& D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{x}, \zeta_2^m)k_0(\underline{x}, \zeta_2^m) \\
& -\delta(w^{-p}\zeta_1/\zeta_2)\frac{d}{d\zeta_1}(K_0(\underline{x}, \zeta_1^m)k_0(\underline{x}, \zeta_2^m))|_{(w^{-p}\zeta_2, \zeta_1)} \\
& = D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{x} + \underline{x}, \zeta_2^m) \\
& +\delta(w^{-p}\zeta_1/\zeta_2)m\sum r_i k_i(\underline{x}, \zeta_1^m)k_0(\underline{x}, \zeta_2^m) \\
& = D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{x} + \underline{x}, \zeta_2^m) \\
& +m\delta(w^{-p}\zeta_1/\zeta_2)\sum r_i k_i(\underline{x} + \underline{x}, \zeta_1^m)
\end{aligned}$$

Thus (5.3) equals to the R.H.S. of (7) of (1.10)

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